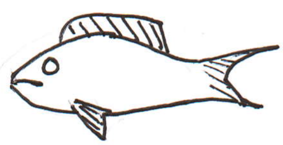
Chain Rule Lecture 1:

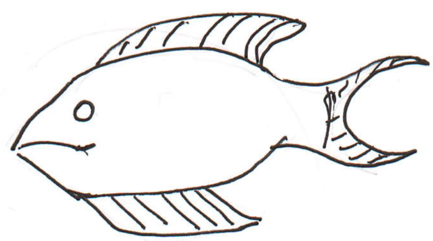
The product and quotient rules deal with differentiating functions of the form $f \cdot g$ and $\frac{f}{g}$. We know for example that $\frac{d}{dx}(x \sin x) = \frac{d}{dx}(x)\sin x + x\frac{d}{dx}(\sin x) =$
$= \sin x + x\cos x$.

Let us take note of the type of functions we know how to differentiate thus far. I like to think of it as taking inventory of the fish in our aquarium.



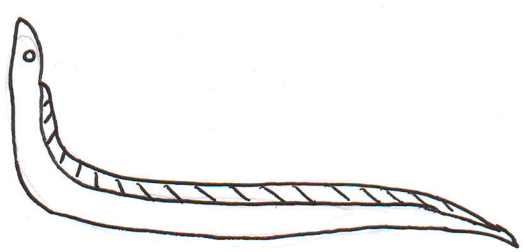
$p(x) = a_0 + a_1 x + \dots + a_n x^n$

Polynomials



$r(x) = \sqrt[n]{x}$

Roots

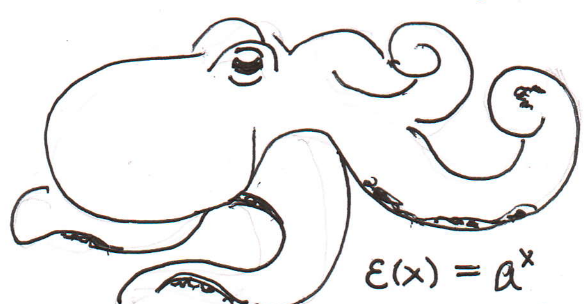


$I(x) = \frac{1}{x}$

Involution

$t(x) = \sin x, \cos x, \dots$

Trigonometric functions



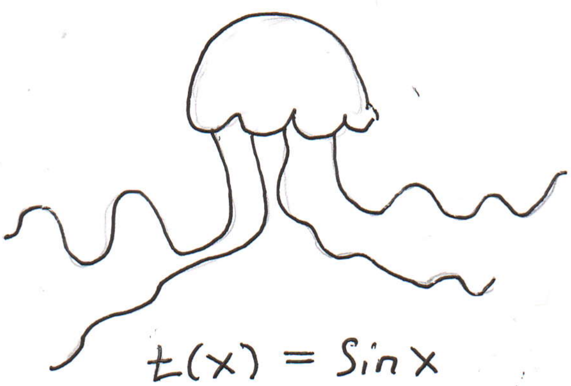
$\varepsilon(x) = a^x$

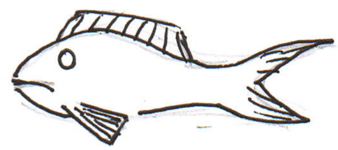
Exponential functions.

I find it helpful to imagine differentiating a function to be analogous to filleting fish.

<u>Ex.</u> Compute $\frac{d}{dx}\left(\sin(x^2+1)\right)$

<u>Solution:</u> First try to think what creature(s) are we dealing with? What fishermen stories can we tell about the catch?





$p(x) = 1+x^2$

What is $\sin(x^2+1)$? Well its a medusa that has eaten a polynomial! When we take the derivative the process is much like:

$$\frac{d}{dx}\left(\sin(x^2+1)\right) = \underbrace{\cos(x^2+1)} \cdot \underbrace{2x}$$

cut open the medusa and find the poly fish in belly

$\frac{d}{dx} \sin\square$
$= \cos\square$

fillet the fish inside that belly

$\frac{d}{dx}(x^2+1) = 2x.$

Ex. Compute $\frac{d}{dx}\left(\sin^2 x + 1\right)$.

Solution: what has happened now? Well, it looks like we are dealing again with medusa trig and polynomial fish! But this time the polynomial fish ate the medusa.

We see $p(x) = x^2 + 1$ and $t(x) = \sin x$

place food in belly of fish (a number)

the digestion process squares "food" and adds 1.

note that an explicit process of digestion is not described!

$$\frac{d}{dx}\left(\sin^2 x + 1\right) = \underbrace{2\boxed{\sin x}}_{\text{cut open outer fish } \square^2+1 \;\mapsto\; 2\square} \cdot \underbrace{\boxed{\cos x}}_{\text{cut open the inner fish } \sin x \;\longmapsto\; \cos x}$$

Ex. For each of the following, tell me a fisherman's story and then differentiate the function.

(a) $\frac{d}{dx} e^{x^2}$

(b) $\frac{d}{dx} \tan(\sqrt{x})$

(c) $\frac{d}{dx} \sec\left(\frac{1}{x}\right)$

(d) $\frac{d}{dx} e^{\csc x}$

Solution:

(a) The octopus e^x has eaten the polynomial fish x^2

$$\frac{d}{dx} e^{x^2} = \underbrace{e^{\boxed{x^2}}}_{\substack{\text{fillet } e^{\square} \\ \mapsto e^{\square}}} \cdot \underbrace{\boxed{2x}}_{\substack{\text{fillet } x^2 \\ \mapsto 2x}}$$

(b) The medusa $\tan x$ has eaten the root fish $\sqrt{x}$

$$\frac{d}{dx} \tan(\sqrt{x}) = \underbrace{\sec^2 \boxed{\sqrt{x}}}_{\substack{\text{fillet } \tan\square \\ \mapsto \sec^2\square}} \cdot \underbrace{\boxed{\frac{1}{2\sqrt{x}}}}_{\substack{\text{fillet } \sqrt{x} \\ \mapsto \frac{1}{2\sqrt{x}}}}$$

(c) The involution eel $\frac{1}{x}$ was eaten by the medusa $\sec x$.

$$\frac{d}{dx} \sec\left(\frac{1}{x}\right) = \underbrace{\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)}_{\substack{\text{fillet } \sec\square \\ \mapsto \sec\square\tan\square}} \cdot \underbrace{\boxed{\frac{-1}{x^2}}}_{\substack{\text{fillet } \frac{1}{x} \\ \mapsto \frac{-1}{x^2}}}$$

(d) The octopus e^x has eaten the medusa $\csc x$.

$$\frac{d}{dx} e^{\csc x} = \underbrace{e^{\boxed{\csc x}}}_{\substack{\text{fillet } e^{\square} \\ \mapsto e^{\square}}} \cdot \underbrace{\boxed{-\csc x \cot x}}_{\substack{\text{fillet } \csc x \\ \mapsto -\csc x \cdot \cot x.}}$$

Why does Chain Rule work?

Thm: (Chain Rule) Let $f(x)$ be differentiable at $x=a$ with derivative $f'(a)$. Let $g(x)$ be differentiable at $x=f(a)$. Then $\frac{d}{dx}\left(g(f(x))\right)\Big|_{x=a} = g'(f(a))\cdot f'(a)$.

Proof: (a) By definition of derivative.

Let $k(x) = g\circ f(x) = g(f(x))$. Then

$$k'(a) = \lim_{x\to a} \frac{k(x)-k(a)}{x-a} = \lim_{x\to a} \frac{g(f(x))-g(f(a))}{x-a}$$

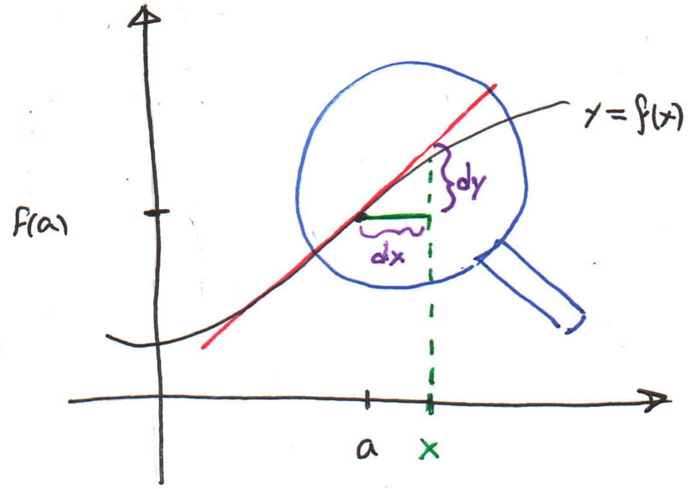
what would you like the denominator to be?

$$= \lim_{x\to a} \frac{g(f(x))-g(f(a))}{\boxed{f(x)-f(a)}}\cdot\frac{\boxed{f(x)-f(a)}}{x-a}$$

$$= \lim_{\substack{y\to f(a)\\ y=f(x)}} \frac{g(y)-g(f(a))}{y-f(a)}\cdot \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = g'(f(a))\cdot f'(a)$$

* $\lim_{x\to a} \frac{g(f(x))-g(f(a))}{f(x)-f(a)}$ looks like the definition of derivative of g at $x=f(a)$.

(b) We can also use our geometric understanding to extract the formula.



Recall: When x is close to a, $f(x)$ is very similar to the line that approximates it near $x = a$.

That is $y(x) = f(a) + f'(a)\overbrace{(x-a)}^{dx}$ and $f(x)$ are very similar!

I call $y(x)$ kackagip or "stunt double".

The stunt double of $g(x)$ near $f(a)$ is

$$V(x) = g(f(a)) + g'(f(a))(x - f(a))$$

Now $g(f(x)) \approx g(f(a)) + g'(f(a))(f(x) - f(a))$

$$\approx g(f(a)) + g'(f(a))(f(a) + f'(a)(x-a)) = g(f(a)) + g'(f(a)) \cdot f'(a)(x-a)$$

$$= g(f(a)) + \underbrace{g'(f(a))f'(a)}_{\text{slope of the line.}}dx$$

Didn't catch it? Instead of g eating f, we let the stunt-double of g eat the stunt-double of f!

Practice

(a) $\frac{d}{dx}\left(4x-x^2\right)^{100}$

(b) $\frac{d}{dx}\left(\sin(e^x)+e^{\sin x}\right)$

(c) $\frac{d}{ds}\left(\sqrt{\frac{s^2+1}{s^2+4}}\right)$

(d) $\frac{d}{dt}\left(\tan(\cos t)\right)$

(e) $\left(\sin(\tan(e^{x^2}))\right)' = ?$

(f) $\left(\sqrt{1+xe^{-2x}}\right)' = ?$

(g) $\left(\sqrt{x+\sqrt{x+\sqrt{x}}}\right)' = ?$

(h) $\left(\sqrt{x+\sqrt{x+\sqrt{x+\cdots+}}}\right)' = ?$

Solution:

(a) $\frac{d}{dx}\left(4x-x^2\right)^{100} = 100\left(4x-x^2\right)^{99}\cdot(4-2x)$

(b) $\frac{d}{dx}\left(\sin(e^x)+e^{\sin x}\right) = \cos(e^x)\cdot e^x + e^{\sin x}\cdot \cos x$

(c) $\frac{d}{ds}\left(\sqrt{\frac{s^2+1}{s^2+4}}\right) = \frac{1}{2\sqrt{\frac{s^2+1}{s^2+4}}}\cdot\frac{(s^2+4)\cdot 2s - 2s(s^2+1)}{(s^2+4)^2}$

$= \frac{1}{2}\sqrt{\frac{s^2+4}{s^2+1}}\cdot\frac{6s}{(s^2+4)^2} = \sqrt{\frac{s^2+4}{s^2+1}}\cdot\frac{3s}{(s^2+4)^2}$

(d) $\frac{d}{dt}\left(\tan(\cos t)\right) = \sec^2(\cos t)\cdot(-\sin t)$

(e) $\left(\sin(\tan(e^{x^2}))\right)' = \cos(\tan(e^{x^2}))\cdot\sec^2(e^{x^2})\cdot e^{x^2}\cdot 2x$.

(f) $\left(\sqrt{1+xe^{-2x}}\right)' = \frac{e^{-2x}-2xe^{-2x}}{2\sqrt{1+xe^{-2x}}}$

(g) $$\left(\sqrt{x+\sqrt{x+\sqrt{x}}}\right)' = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot$$

$$\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)\right)$$

(h) This is actually easier than (g)!

Let $f(x) = \sqrt{x+\underbrace{\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}}_{f(x)}} = \sqrt{x+f(x)}$

Hence $f^2(x) = x + f(x)$

$f^2(x) - f(x) - x = 0 \Longrightarrow f(x) = \dfrac{1+\sqrt{1+4x}}{2}$

so $f'(x) = \left(\frac{1}{2}+\frac{1}{2}\sqrt{1+4x}\right)' = \frac{1}{4}\frac{1}{\sqrt{1+4x}}\cdot 4 = \dfrac{1}{\sqrt{1+4x}}$.